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The Transient Hot-Wire Technique: A Numerical Approach[1]

M. J. Assael,[2,3] L. Karagiannidis,[2] N. Malamataris,[2] and W. A. Wakeham[4]

The measurement of the thermal conductivity of a fluid by means of the transient hot-wire technique so far has made use of an analytical solution of the energy conservation equation for an ideal model, coupled with a set of approximate analytical corrections to account for small departures from the model. For this solution to be valid, constraints were always imposed on the experimental conditions and the construction of the apparatus, resulting in an inability to measure the thermal conductivity of high-thermal diffusivity fluids. In this paper, the set of energy conservation equations describing the transient hot-wire apparatus is solved using the numerical finite-element method. Because no approximate solutions are involved, this provides a much more general treatment of the heat transfer processes taking part in the real experiment, removing all the aforementioned constraints. In the case of the measurement of the thermal conductivity of liquids (fluids with low thermal-diffusivity values), the numerical solution fully agrees with the existing analytical solution. In the case of the measurement of the thermal conductivity of gases, the present solution allows the extension of the application of the transient hot-wire technique to experimental conditions where the value of the thermal diffusivity of the fluid is high.

KEY WORDS: energy conservation equations; finite-element method; high-thermal diffusivity fluids; low-density measurements; thermal conductivity; transient hot-wire.

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[2] Department of Chemical Engineering, Aristotle University of Thessaloniki, 54006 Thessaloniki, Greece.

[3] To whom correspondence should be addressed.

[4] Department of Chemical Engineering and Chemical Technology, Imperial College of Science, Technology and Medicine, London SW7 2BY, United Kingdom.

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1. INTRODUCTION

The transient hot-wire technique is widely employed today for the accurate measurement of the thermal conductivity of fluids, in both the liquid and the gas phases, over a wide range of temperatures and pressures. However, there are two regions of the thermodynamic state where the application of the method presents a loss of accuracy. The first region is near the critical point and the second, in the region where fluids exhibit high thermal diffusivity values, i.e., the low-density region. The observed inability [1] of thermal conductivity measurements in the latter region is very unfortunate, as accurate zero-density thermal conductivity values are required by kinetic theory treatments. Moreover, as has been stated elsewhere [2], the same problem appears in the measurement of the thermal conductivity of refrigerants in the vapor phase, since these fluids generally show a maximum pressure of 2 MPa along the saturation line. Previous studies that attributed the observed inability to the compression work of the gas were rejected, since a recent analytical and numerical study [3] has proved that the effect of compressibility is always insignificant in the dilute-gas region.

In this paper, the energy conservation partial-differential set of equations that describe the transient hot-wire theory is solved using the numerical finite-element method. A computer program was developed and employed in order to study the evolving temperature field within the wire and the fluid. From the comparison of the measured temperature rise and the simulated one, the thermal conductivity of argon was obtained at high and low pressures, in a transient hot-wire instrument. To conclude the analysis, this numerical solution is compared with the analytical one employing (a) the exact corrections given by Carslaw and Jaeger [4] and (b) the approximate corrections given by Healy et al. [5].

2. THEORY OF THE TRANSIENT HOT-WIRE TECHNIQUE

2.1. The Analytical Treatment

The detailed theory of the transient hot-wire technique is described elsewhere [6]. The analytical working equation of the method is [6]:

$$\Delta T(r, t) = \frac{q}{4\pi\lambda}\left[\ln\left(\frac{4\kappa t}{r^2 \mathrm{C}}\right) + \left(\frac{r^2}{4\kappa t}\right) + \cdots\right] \tag{1}$$

where $\Delta T(r, t)$ is the transient temperature rise at a radial distance r in the fluid, λ and κ are the thermal conductivity and thermal diffusivity of the

fluid, respectively, q is the heat input power per unit length, and C is the Euler's constant. Equation (1) describes the ideal model which cannot be employed directly to the practical instrument. A number of analytical corrections were developed [4] and imposed on Eq. (1) to eliminate the departure of a practical instrument from the ideal one. The two major applied corrections are as follows.

(a) The heat capacity correction, ΔT_{hc}, significant only at short experimental times [4]:

$$\Delta T = \frac{2q^{1/2}}{p^2\alpha^2} \int_0^\infty (1 - e^{-\kappa_w u^2 t})\, J_1(ua) \times [J_0(\sqrt{\kappa_w/\kappa}\, ur)\, \phi(u) - Y_0(\sqrt{\kappa_w/\kappa}\, ur)\, \phi(u)] \times \{u^3[\phi^2(u) + \psi^2(u)]\}^{-1}\, du \tag{2}$$

with

$$\phi(u) = \lambda_w \kappa^{1/2} J_1(ua)\, J_0(\sqrt{\kappa_w/\kappa}\, ua) - \lambda \kappa_w^{1/2} J_0(ua)\, J_1(\sqrt{\kappa_w/\kappa}\, ua) \tag{3}$$

$$\psi(u) = \lambda_w \kappa^{1/2} J_1(ua)\, Y_0(\sqrt{\kappa_w/\kappa}\, ua) - \lambda \kappa_w^{1/2} J_0(ua)\, Y_1(\sqrt{\kappa_w/\kappa}\, ua) \tag{4}$$

In these expressions, α is the wire radius and the subscript "w" refers to wire properties. Furthermore, J_0 and J_1 denote Bessel functions of the first kind, of order zero and one, respectively, while Y_0 and Y_1 express Bessel functions of the second kind, of order zero and one.

(b) The outer boundary correction, ΔT_{ob}, significant only at long experimental times:

$$\Delta T_{\mathrm{ob}}(a, t) = \frac{q}{2\pi\lambda} \ln \frac{b}{a} + \frac{q}{2a\lambda} \sum_{n=1}^{\infty} e^{-\kappa a_n^2 t} \times \frac{J_0^2(ba_n)[J_0(aa_n)\, Y_1(aa_n) - Y_0(aa_n)\, J_1(aa_n)]}{a_n[J_1^2(aa_n) - J_0^2(ba_n)]} \tag{5}$$

where b is the wire-enclosure radius and a_n are the positive roots of the equation

$$J_1(ax)\, Y_0(bx) - Y_1(ax)\, J_0(bx) = 0 \tag{6}$$

As can be seen from Eqs. (2)–(6), these corrections not only are mathematically too complicated but also cannot be calculated analytically.

Healy et al. [5] proposed approximate solutions of Eqs. (2) and (5) valid for large values of $(4\kappa t/a^2)$. These solutions together with Eq. (1) formed a consistent set in order to calculate the thermal conductivity from the measured temperature rise. The application of this methodology to liquids and gases at moderate pressures has provided many reliable thermal conductivity data over the last two decades. Unfortunately, the corrections proposed by Healy et al. [5] proved to be inadequate [1, 7] for the description of experimental runs at low densities, where fluids exhibit high thermal diffusivity values. Consequently, until now measurements in this region were avoided.

2.2. The Numerical Solution

The inability of the Healy methodology to describe low-density experimental thermal conductivity data motivated us to apply a different approach for the calculation of thermal conductivity from the measured temperature rise. Hence, the numerical finite-element method (FEM) was chosen in order to solve the complete set of energy conservation equations that describe the heat transfer experimental processes. The choise of this particular numerical method was dictated by the high accuracy the method exhibits in computational heat transfer problems [8].

The energy equations to be solved are two coupled partial differential equations, one for the wire, $0 < r \leqslant \alpha$,

$$(\rho C_p)_w \frac{\partial T_w}{\partial t} = \lambda_w \left(\frac{1}{r} \frac{\partial T_w}{\partial r} + \frac{\partial^2 T_w}{\partial r^2} \right) + q/\pi\alpha^2 \tag{7}$$

and one for the fluid, $\alpha \leqslant r < \infty$,

$$(\rho C_p) \frac{\partial T}{\partial t} = \lambda \left(\frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial r^2} \right) \tag{8}$$

The symbols ρ and C_p denote the density and the specific heat. On the wire/fluid interface both the temperature and the heat flux are considered to be continuous. This means that for $r = \alpha$,

$$\lambda_w \left(\frac{\partial T_w}{\partial r} \right)_{r=a} = \lambda \left(\frac{\partial T}{\partial r} \right)_{r=a} \qquad \text{and} \qquad T_w(a, t) = T(a, t) \tag{9}$$

The aforementioned set of equations is subject to the following initial and boundary conditions.

For $t=0$ and any r, $$T_w = T = 0 \tag{10}$$

For $t>0$ and at $r=0$, due to symmetry, $$\frac{\partial T_w}{\partial r} = 0 \tag{11}$$

For $t>0$ and at $r=b$, $$T = 0 \tag{12}$$

Equations (7)–(12) are written in a dimensionless form using the following dimensionless variables:

$$r^* = \frac{r}{a}, \qquad t^* = \frac{\kappa_w t}{a^2}, \qquad T_w^* = \frac{\lambda_w T_w}{q}, \qquad \text{and} \qquad T^* = \frac{\lambda_w T}{q} \tag{13}$$

Substituting the aforementioned dimensionless variables in Eqs. (7)–(12), the following set of dimensionless partial differential equations and initial and boundary conditions is obtained:

$$\frac{\partial T_w^*}{\partial t^*} = \left[\frac{1}{r^*}\left(\frac{\partial T_w^*}{\partial r^*}\right) + \frac{\partial^2 T_w^*}{\partial r^{*2}}\right] + \frac{1}{\pi} \tag{14}$$

$$\left[\frac{\rho C_p}{(\rho C_p)_w}\right]\frac{\partial T^*}{\partial t^*} = \frac{\lambda}{\lambda_w}\left[\frac{1}{r^*}\left(\frac{\partial T^*}{\partial r^*}\right) + \frac{\partial^2 T^*}{\partial r^{*2}}\right] \tag{15}$$

For $t^*=0$ at any r^*, $$T_w^* = T = 0 \tag{16}$$

For $t^*>0$ and $r^*=0$, $$\frac{\partial T_w^*}{\partial r^*} = 0 \tag{17}$$

For $t^*>0$ and $r^*=b/a$, $$T^* = 0 \tag{18}$$

For $t^*>0$ and $r^*=1$, $$\lambda_w\left(\frac{\partial T_w}{\partial r}\right)_{r=a} = \lambda\left(\frac{\partial T}{\partial r}\right)_{r=a}$$
$$\text{and } T_w^*(a,t) = T^*(a,t) \tag{19}$$

As can be seen from Eqs. (14) and (15), the problem is one-dimensional with respect to radial direction. Therefore, the field of application is a straight line, discretized in 250 finite elements, representing the 6-mm actual radial distance, irregularly spaced. The mesh constructed is a lot more dense at the wire/fluid interface. At each nodal point, the above equations are solved to calculate the value of the temperature rise. Since this is a time-dependent problem, the solution is obtained using the forward difference (Euler) scheme [8]. Fortunately, the set of energy conservation equations described above is linear. As a result, the derived set of algebraic equations is also linear and is solved using a modification of the Gaussian

elimination method (LU decomposition method) [9]. Hence, a computer program was developed for the simulation of the measured temperature rise and the consequent calculation of the thermal conductivity.

2.3. Case Studies

In order to check that the program is working properly and accurately, it was first tested against a problem on conduction of heat in composite solids, very similar to the transient hot-wire case, with a known analytical solution [4]. The numerical solution of the temperature rise simulates the analytical one within $\pm 0.2\%$.

Consequently, the developed algorithm was tested against a transient hot-wire experiment with toluene as the sample fluid. The measured transient temperature rise is compared against the computed temperature rise in Table I. At this stage, the following three points need to be mentioned.

Table I. Comparison between Experimental and Computed (FEM) Temperature Rise for Toluene

Time (s)	Temperature rise (K)		Difference (mK)
	Experimental	Computed	
0.1997	2.321	2.278	43
0.2085	2.343	2.303	40
0.2334	2.388	2.350	38
0.2605	2.433	2.394	39
0.2884	2.478	2.437	41
0.3212	2.523	2.482	41
0.3603	2.568	2.530	38
0.3989	2.613	2.573	40
0.4214	2.635	2.595	40
0.4718	2.680	2.641	39
0.5210	2.726	2.685	41
0.5518	2.748	2.709	39
0.6162	2.793	2.752	41
0.6475	2.816	2.777	39
0.6807	2.838	2.797	41
0.7211	2.862	2.820	42
0.7648	2.884	2.846	38
0.8087	2.906	2.866	40
0.8550	2.929	2.889	40
0.8953	2.952	2.913	39
0.9480	2.974	2.933	41

(a) Since the simulation program solves an energy conservation equation within the metallic wire, it takes into account the properties and the finite dimensions of the wire. Thus, the heat capacity correction is fully taken into consideration.

(b) The temperature distribution function within the radius of the wire can be calculated. From the integration of the temperature distribution function over the wire radius, the value of the transient temperature rise is calculated for the whole wire and not just at the wire/fluid interface.

(c) Finally, the imposition of boundary conditions at the wire enclosure circumference, assumes the existence of a real isothermal wall at a finite radius and not at infinity and, therefore, the outer boundary correction is also properly accounted for.

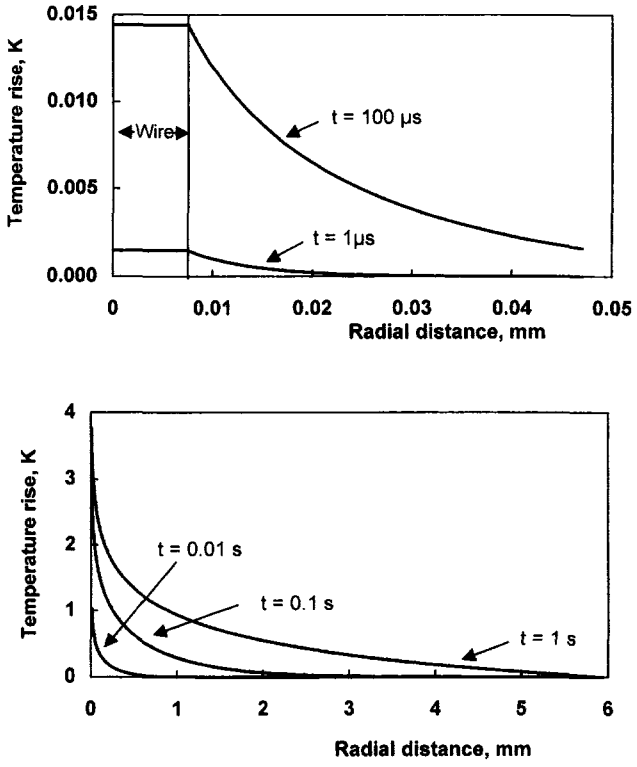

Fig. 1. The temperature rise evolution in the wire and the fluid at different times and axial distances.

When examining Table I, one must keep in mind that the thermal conductivity is calculated from the slope of the temperature rise vs the logarithm of the time. Therefore absolute differences in the temperature rise (about 40 mK) are insignificant in relation to the difference in slopes, which is only ± 2 mK. Thus, the comparison shows that the slopes of the simulated and the experimental temperature rises are identical. Hence, the correct value of the thermal conductivity is calculated using the FEM for the theoretical analysis of the experimental data. More experimental runs on liquid and gases at elevated pressures were checked against the computer program and the thermal conductivity computed at all times was accurately obtained. It is therefore concluded that the FEM in the aforementioned cases produces thermal conductivity values at least as accurate as those produced by the approximate analytical solution. Furthermore, the implementation of the numerical method has the advantages of an easy check of all other parameters that may influence the actual experiment and are included in the energy conservation equations. An example of the temperature evolution in the wire and the fluid is shown in Fig. 1.

3. EXPERIMENTAL PROCEDURE

The high accuracy attained in the calculation of the thermal conductivity values using the numerical FEM has led us to its application in the analysis of thermal-conductivity measurements of fluids with high thermal diffusivity values, e.g., argon at room temperature and atmospheric pressure. Therefore, a series of measurements was performed in a transient hot-wire instrument especially constructed for measurements of the thermal conductivity of refrigerants in the vapor phase. The full description of the instrument is given in detail elsewhere [2]. Transient measurements were performed in argon at 300 K and pressures from 0.15 up to 4.8 MPa. The experimental temperature rise data were analyzed applying three methodologies.

(A) The analytical solution with approximate corrections proposed by Healy et al. [5].

(B) The analytical solution with exact corrections proposed by Carslaw and Jaeger [4]. As these corrections cannot also be calculated analytically, the 3/4 Simpson's numerical integration technique was adopted for the calculation of Eqs. (2) and (5). In the case of the outer boundary correction, 20 positive roots of Eq. (6) were found to be enough for the accurate calculation of the correction.

(C) The numerical solution using the FEM.

Table II. Thermal Conductivity Values ($mW \cdot m^{-1} \cdot K^{-1}$) of Argon at 300 K at Different Pressures

Pressure (MPa)	Analytical solution + Healy et al.'s [5] approximate corrections	Analytical solution + Carslaw and Jaeger's [4] corrections	FEM numerical solution	Literature value [10]
4.80	19.80	19.84	19.87	19.80
3.40	19.10	19.21	19.20	19.16
2.60	18.80	18.78	18.79	18.84
1.50	18.32	18.34	18.36	18.39
1.05	18.10	18.15	18.14	18.21
0.35	17.55	17.90	17.92	17.93
0.14	17.30	17.72	17.73	17.83

The results obtained using these three methodologies are shown in Table II. In the same table, for comparison purposes, the "correct" values of the thermal conductivity of argon [10] are also shown. The thermal conductivity values calculated by the numerical FEM and the analytical solution using the exact corrections proposed by Carslaw and Jaeger [4] agree very well both mutually and with the literature values. The observed difference, in Table II, between the thermal conductivity values calculated by the

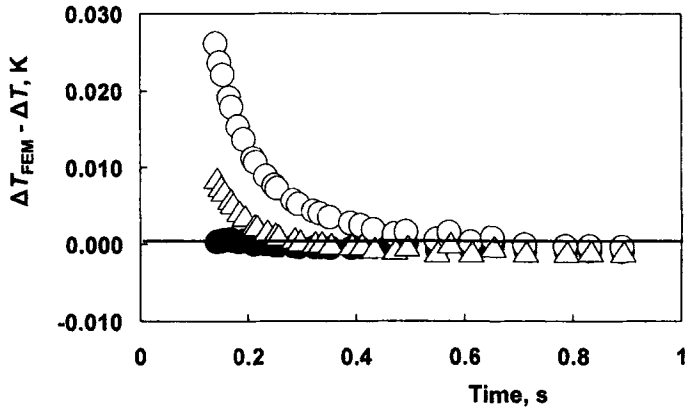

Fig. 2. Deviations from the temperature rise calculated by the FEM approach, of (a) (●) the ideal solution incorporating Carslaw and Jaeger corrections [4], (b) (○) the Healy et al. equations [5], and (c) (△) the Taxis and Stephan approximate equations [7], for argon at 300 K and 0.15 MPa, for an isothermal wall at a 6-mm radial distance from the wire.

analytical solution with the corrections proposed by Healy et al. [5] and the values calculated by the other two methodologies are attributed in general to the approximations inserted by Healy et al., in order to simplify both the heat capacity and the outer boundary corrections.

The advantage of the FEM can be clearly seen in the treatment of the heat capacity correction. Although the outer boundary correction can be rendered very small by chosing a large diameter enclosure, the heat capacity correction still must be calculated. In Fig. 2 we show a comparison of the two aforementioned approaches from the FEM solution, for an experimental run of argon at 300 K and 0.15 MPa. The numerical FEM and the analytical solution using the exact corrections proposed by Carslaw and Jaeger [4] are very similar as expected. An approximate solution given by Taxis and Stephan [7] is also included in Fig. 2.

4. CONCLUSIONS

Based on the FEM a computer program was successfully developed that is able to solve the energy conservation equations that describe the heat transfer processes, taking place during an experimental run in a transient hot-wire instrument. Comparisons between the different existing methodologies employed to compute the thermal conductivity from experimental data revealed differences, especially for gases in the low pressure region. In this region both the heat capacity and the outer boundary corrections are significant. Therefore, for accurate theoretical analysis of the experimental data produced by the transient hot-wire technique, either the complete set of corrections proposed by Carslaw and Jaeger [4] must be employed or, better, a numerical FEM. For the outer boundary correction, either the wall enclosure must be moved away from the wire or, again, a FEM approach should be employed. The FEM calculates all the significant corrections to the transient hot-wire ideal model, avoiding extra simplification of the working equations to analytical forms that cannot simulate accurately the experimentally measured temperature rise, especially for the case of fluids with high thermal diffusivity values.

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